McKean-Vlasov SDEs

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Mean field CKLS model

McKean-Vlasov SDEs with Hölder Continuous drift

Idea of Proof

Mean Field CKLS Model

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Outline

♣ McKean-Vlasov SDEs

♣ Well-posedness for SDEs with Hölder continuous and distribution dependent diffusion

♣ Mean field CKLS model

♣ Log-Harnack inequality

♣ Exponential ergodicity in Wasserstein distance

♣ Idea of Proofs

McKean-Vlasov SDEs

- $\mathscr{P}(\mathbb{R}^d)$-all probability measures on $\mathbb{R}^d$.
- For $p > 0$, let

$$\mathscr{P}_p(\mathbb{R}^d) = \{\mu \in \mathscr{P}(\mathbb{R}^d), \mu(|\cdot|^p) := \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) < \infty\}.$$

- $\mathbb{W}_p$-Wasserstein distance:

$$\mathbb{W}_p(\mu,\nu) = \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{1 \vee p}}, \ \ \mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$$

  $\mathcal{C}(\mu,\nu)$-all couplings of μ and ν.
- DDSDEs

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t, \ \ t \ge 0,$$

McKean-Vlasov SDEs

- Existence and Uniqueness. M. Röckner, X. Zhang; G, Zhao; D. Lacker; P.-E. Chaudru de Raynal; Y. S. Mishura, A. Yu Veretennikov; J. Bao, X. Huang, F.-Y. Wang and so on.
- Distribution property (Harnack inequality, Bismut formula, Estimate of L-derivative), C. Deng; Y. Song; P. Ren; F.-Y. Wang;
- Order preservation: X. Huang, Feng-Yu Wang, Chenggui Yuan.
- Correspondence between nonlinear Fokker-Planck-Kolmogorov equations and DDSDEs: V. Barbu, M. Röckner;
- Ergodicity: W. Liu, L. Wu; J. Wang; P. Ren, F.-Y. Wang;
- Large deviation: W. Liu, Y. Song, J. Zhai; T. Zhang;
- Numerical scheme: X. Zhang; J. Bao.
- Phase transform: M.-F. Chen; S.-Q. Zhang.

Mean field method in Markov Chain

- Birth and death process,

$$q_{i,i+1} = b_i, i \geq 0, q_{i,i-1} = a_i, i \geq 1, q_{i,i} = -(b_i + a_i)$$

- Generator

$$\Omega f(i) = b_i(f(i+1) - f(i)) - a_i(f(i) - f(i-1))$$

- Invariant probability measure

$$\mu_k = \frac{\prod_{i=0}^{k-1} b_i}{\prod_{i=1}^{k} a_i}\mu_0 =: Z_k\mu_0, \quad \sum_{k=0}^{\infty} Z_k < \infty.$$

- Generator

$$\tilde{\Omega}_t f(i) = b_i(f(i+1)-f(i))-a_i(f(i)-f(i-1))+\mathbb{E}(X_t)(f(i+1)-f(i))$$

- $\tilde{b}_i = b_i + \tilde{\mu}(\cdot)$, $\tilde{a}_i = a_i$.

$$\tilde{\mu}_k = \frac{\prod_{i=0}^{k-1} (b_i + \tilde{\mu}(\cdot))}{\prod_{i=1}^{k} a_i}\mu_0 =: \tilde{Z}_k\mu_0, \quad \sum_{k=0}^{\infty} \tilde{Z}_k < \infty.$$

SDEs with Hölder continuous and distribution dependent diffusion

- Consider one-dimensional SDE

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t,\ \ t \geq 0, \tag{2.1}$$

- For some $\theta \in [\frac{1}{2}, 1)$,

$$|\sigma(x,\mu) - \sigma(y,\nu)| \leq K_0|x-y|^{\theta} + K_1\mathbb{W}_1(\mu,\nu)$$

- Note that

$$\int_{\frac{\varepsilon}{\mathrm{e}}}^{\varepsilon} \frac{1}{x} = 1$$

- Yamada-Watanabe approximation

$$\psi_\varepsilon(x) \leq \frac{2}{x}1_{[\frac{\varepsilon}{\mathrm{e}},\varepsilon]},\quad \int_{\frac{\varepsilon}{\mathrm{e}}}^{\varepsilon} \psi_\varepsilon(x) = 1,\quad V_\varepsilon(x) = \int_0^{|x|}\int_0^y \psi_\varepsilon(z)\mathrm{d}z.$$

- Ito's formula

$$V_\varepsilon(X_t - Y_t)$$

- Difficulty:

$$\frac{|\sigma_t(X_t, \mathscr{L}_{X_t}) - \sigma_t(Y_t, \mathscr{L}_{Y_t})|^2}{|X_t - Y_t|}1_{|X_t - Y_t| \in [\frac{\varepsilon}{\mathrm{e}},\varepsilon]} \to \frac{K_1^2\mathbb{W}_1(\mathscr{L}_{X_t}, \mathscr{L}_{Y_t})^2}{|X_t - Y_t|}$$

Well-posedness

- Consider

$$dX_t = (\alpha - \delta X_t)dt + h(\mathbb{E}(X_t))dt + \sigma_t(X_t, \mathbb{E}(X_t))dW_t,\ t \geq 0,\ (2.2)$$

Definition

A continuous adapted process $(X_t)_{t\geq 0}$ is called a strong solution to (2.2) if $\mathbb{E}(X_t)$ is continuous in t, and $\mathbb{P}$-a.s.

$$X_s = X_0 + \int_0^s (\alpha - \delta X_t)dt + \int_0^s h(\mathbb{E}(X_t))dt + \int_0^s \sigma(X_t, \mathbb{E}(X_t))dW_t, \quad s \geq 0$$

We make the following assumptions.

(C) σ is measurable. There exists a constant $L > 0$ and $\theta \in [\frac{1}{2}, 1]$ such that

$$|h(y) - h(\bar{y})| \leq L|y - \bar{y}|, \quad |\sigma(x,y) - \sigma(\bar{x},y)| \leq L|x - \bar{x}|^\theta, \quad x, \bar{x}, y, \bar{y} \in \mathbb{R}$$

and

$$|\sigma(0,y)| \leq L(1 + |y|), \quad y \in \mathbb{R}.$$

Well-posedness

Theorem

Assume (**C**)*. Then for any* $X_0 \in L^1(\Omega, \mathscr{F}_0, \mathbb{P})$, (2.2) *has a unique strong solution* X_t *with initial value* X_0*.*

Proof.

For any $\Gamma \in C([0,T];\mathbb{R})$, define $\sigma_t^\Gamma(x) = \sigma(x, \Gamma_t)$, $b_t^\Gamma(x) = \alpha - \delta x + h(\Gamma_t)$ and

$$\mathrm{d}X_t = b_t^\Gamma(X_t)\mathrm{d}t + \sigma_t^\Gamma(X_t)\mathrm{d}W_t. \tag{2.3}$$

Define $\Phi^\gamma : C([0,T];\mathbb{R}) \to C([0,T];\mathbb{R})$ as

$$\Phi_t^\gamma(\Gamma) = \mathbb{E}(X_t^\gamma(\Gamma)), \quad t \in [0,T].$$

Define $E_0 = \{\Gamma \in C([0,T],\mathbb{R}), \Gamma_0 = \mathbb{E}(X_0)\}$, equipped with

$$d_\lambda(\Gamma, \tilde\Gamma) = \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}|\Gamma_t - \tilde\Gamma_t|, \quad \Gamma, \tilde\Gamma \in E_0.$$

Mean field CKLS model

- Consider

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + \gamma\mathbb{E}(X_t)\mathrm{d}t + |X_t|^\theta \mathrm{d}W_t. \quad (3.1)$$

- For any $\mu_0 \in \mathscr{P}_1$, let $P_t^*\mu_0 = \mathscr{L}_{X_t^{\mu_0}}$. Define

$$P_t f(\mu_0) = \int_{\mathbb{R}} f(x)(P_t^*\mu_0)(\mathrm{d}x), \quad \mu_0 \in \mathscr{P}_1, t \geq 0, f \in \mathscr{B}_b([0,\infty)).$$

- Intrinsic distance

$$\rho(s,t) = \int_{s\wedge t}^{s\vee t} \frac{\mathrm{d}r}{r^\theta} = \frac{(s\vee t)^{1-\theta} - (s\wedge t)^{1-\theta}}{1-\theta}, \quad s,t \in [0,\infty). \quad (3.2)$$

Define

$$\mathbb{W}_{2,\rho}(\mu,\nu)^2 = \inf_{\pi\in\mathscr{C}(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} \rho(x,y)^2 \pi(\mathrm{d}x,\mathrm{d}y)$$

Log-Harnack inequality

Theorem

Assume $\delta > 0$ and $\gamma \geq 0$. Then the following assertions hold.

(1) *Assume $\frac{1}{2} < \theta < 1$ and $\alpha \geq \frac{\theta}{2}$. For any $f \in \mathscr{B}_b^+([0,\infty))$ with $f > 0$, $\mu_0, \nu_0 \in \mathscr{P}_1^+$ with $\max(\mu_0((\cdot)^{1-2\theta}), \nu_0((\cdot)^{1-2\theta})) < \infty$, the log-Harnack inequality holds, i.e.*

$$P_T \log f(\mu_0) \leq \log P_T f(\nu_0) + \frac{(1-\theta)(\delta - \frac{\theta}{2})\mathbb{W}_{2,\rho}(\mu_0,\nu_0)^2}{(\mathrm{e}^{2(1-\theta)(\delta-\frac{\theta}{2})T} - 1)} + \frac{1}{2}\gamma^2\Gamma\mathbb{W}_1(\mu_0,\nu_0)^2$$

for some constant Γ depending on T,γ,δ, α, θ $\mu_0(\cdot)$, $\mu_0((\cdot)^{1-2\theta})$, $\nu_0(\cdot)$, $\nu_0((\cdot)^{1-2\theta})$.

Log-Harnack inequality

Theorem

(2) *Assume* $\theta = \frac{1}{2}$ *and* $\alpha > \frac{1}{2}$. *Then for any* $f \in \mathscr{B}_b^+([0,\infty))$ *with* $f > 0$, $\mu_0, \nu_0 \in \mathscr{P}_1^+$ *satisfying* $\max(\mu_0(|\log(\cdot)|), \nu_0(|\log(\cdot)|)) < \infty$, *the log-Harnack inequality holds, i.e.*

$$P_T \log f(\mu_0) \leq \log P_T f(\nu_0) + \frac{\frac{1}{2}(\delta - \frac{1}{4})\mathbb{W}_{2,\rho}(\mu_0,\nu_0)^2}{\frac{1}{2}(\mathrm{e}^{(\delta-\frac{1}{4})T} - 1)} + \frac{1}{2}\gamma^2\bar{\Gamma}\mathbb{W}_1(\mu_0,\nu_0)^2.$$

for some constant $\bar{\Gamma}$ *depending on* T,γ,δ, α, θ $\mu_0(\cdot)$, $\mu_0(|\log(\cdot)|)$, $\nu_0(\cdot)$, $\nu_0(|\log(\cdot)|)$.

Nonnegative Solution

Lemma

Assume $\alpha, \gamma \geq 0, \delta \in \mathbb{R}$. *Let* X_t *be the solution to* (3.1) *with* $\mathscr{F}_0$-*measurable non-negative initial value* X_0. *Then* $\mathbb{P}$-*a.s.* $X_t \geq 0, \quad t \geq 0$

Proof.

Define (See X. Mao, T. Aubrey, C. Yuan.)

$$\bar{V}_\varepsilon(x) = \int_0^{x^-} \int_0^y \psi_\varepsilon(z) \mathrm{d}z \mathrm{d}y.$$

It is not difficult to see that

$$V_\varepsilon^0(x) = 0, x \geq -\varepsilon/\mathrm{e}, \quad x^- - \varepsilon \leq V_\varepsilon^0(x) \leq x^-, \quad x \in \mathbb{R}, \tag{3.3}$$

$$(V_\varepsilon^0)'(x) \in [-1, 0], \quad x \leq -\varepsilon/\mathrm{e}, \quad (V_\varepsilon^0)'(x) = 0, \quad x \geq -\varepsilon/\mathrm{e}, \tag{3.4}$$

$$0 \leq (V_\varepsilon^0)''(x) \leq 2/x^- \mathbf{1}_{[\varepsilon/\mathrm{e}, \varepsilon]}(x^-), \qquad x \in \mathbb{R}. \tag{3.5}$$

Strictly Positive Solution

Let $\zeta \in C([0,\infty);[0,\infty))$. For $\alpha, \gamma \geq 0, \delta \in \mathbb{R}$, consider

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + \gamma \zeta_t \mathrm{d}t + X_t^{\theta} \mathrm{d}W_t. \tag{3.6}$$

Repeating the proof of this Lemma by replacing $\mathbb{E}(X_t)$ with ζ_t, (3.6) has a unique nonnegative solution X_t^{ζ} with nonnegative initial value.

Lemma

Assume $\alpha, \gamma \geq 0, \delta \in \mathbb{R}$.

(1) *Assume* $\theta \in (\frac{1}{2}, 1)$. *Then for any* $X_0 > 0$, $\mathbb{P}$*-a.s.* $X_t > 0, \quad t \geq 0$. *If moreover* $\delta > 0$ *and* $\mathbb{E}X_0 + \mathbb{E}X_0^{1-2\theta} < \infty$, *it holds*

$$\mathbb{E}\int_0^T (X_t^{\zeta})^{-2\theta} \mathrm{d}t \leq \Gamma_0.$$

(2) *Assume* $\theta = \frac{1}{2}$ *and* $\alpha > \frac{1}{2}$, *for any* $X_0 > 0$, $\mathbb{P}$*-a.s.* $X_t > 0, \quad t \geq 0$. *If moreover,* $\delta > 0$ *and* $\mathbb{E}X_0 + \mathbb{E}|\log(X_0)| < \infty$, *we obtain*

$$\mathbb{E}\int_0^T (X_t^{\zeta})^{-1} \mathrm{d}t \leq \bar{\Gamma}_0.$$

Strictly Positive Solution

Proof.

For any $n \geq 1$, define

$$\tau_n = \inf\{t \geq 0 : X_t \leq \frac{1}{n}\} \wedge \inf\{t \geq 0 : X_t \geq n\}$$

τ_n is increasing in n and it is sufficient to prove $\lim_{n\to\infty} \tau_n = \infty$.

(1)

$$V(x) = x + \frac{1}{2\theta - 1} x^{1-2\theta} - (1 + \frac{1}{2\theta - 1}), \quad x > 0.$$

(2)

$$\bar{V}(x) = x - \log x - 1, \quad x > 0.$$

□

Log-Harnack inequality for classical SDEs

Consider SDE

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t - \delta X_t \mathrm{d}t + X_t^\theta \mathrm{d}W_t, \tag{3.7}$$

here $\alpha : [0,\infty) \to \mathbb{R}$. Let $X_t^{\alpha,\mu}$ be the solution to (3.7) from initial distribution μ and define

$$P_t^\alpha f(x) = \mathbb{E} f(X_t^{\alpha,\delta_x}), \quad t \geq 0, f \in \mathscr{B}_b(\mathbb{R}).$$

By repeating the proof of [10, Theorem 1.2], we can get

Lemma

Let $\frac{1}{2} \leq \theta < 1$ and $\alpha_t \geq \frac{\theta}{2}, t \geq 0$. Then for any x, $y \in [0,\infty)$, we have

$$P_T^\alpha \log f(y) \leq \log P_T^\alpha f(x) + \frac{(1-\theta)(\delta - \frac{\theta}{2})\rho(x,y)^2}{(\mathrm{e}^{2(1-\theta)(\delta-\frac{\theta}{2})T} - 1)}.$$

Therefore, it holds

$$\mathbb{E}(\log f)(X_T^{\mu_0}) \leq \log \mathbb{E} f(X_T^{\nu_0}) + \frac{(1-\theta)(\delta - \frac{\theta}{2})\mathbb{W}_{2,\rho}(\mu_0,\nu_0)^2}{(\mathrm{e}^{2(1-\theta)(\delta-\frac{\theta}{2})T} - 1)}, \quad \mu_0, \nu_0 \in \mathscr{P}_1^+,$$

Exponential Ergodicity

Theorem

Assume $\delta > \gamma$. Then (3.1) has a unique invariant probability measure μ satisfying

$$\mathbb{W}_1(P_t^*\mu_0, \mu) \leq \mathrm{e}^{-(\delta-\gamma)t}\mathbb{W}_1(\mu_0, \mu), \quad \mu_0 \in \mathscr{P}_1.$$

Yamada-Watanabe approximation to get

$$\mathbb{E}|X_s - Y_s| \leq \mathrm{e}^{-(\delta-\gamma)s}\mathbb{E}|X_0 - Y_0|.$$

Idea of Proof-Log-Harnack inequality

- Let $\mu_t = P_t^* \mu_0, \nu_t = P_t^* \nu_0$. Consider

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + \gamma \mu_t(\cdot)\mathrm{d}t + X_t^{\theta}\mathrm{d}W_t.$$

We rewrite this equation as

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + \gamma \nu_t(\cdot)\mathrm{d}t + X_t^{\theta}\mathrm{d}\tilde{W}_t,$$

here

$$\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + X_t^{-\theta}(\gamma\mu_t(\cdot) - \gamma\nu_t(\cdot)).$$

For any $n \geq 1$, define

$$\tau_n = \inf\{t \geq 0 : X_t \leq \frac{1}{n}\}.$$

Let

$$R_s = \exp\left\{ - \int_0^s X_t^{-\theta}(\gamma\mu_t(\cdot) - \gamma\nu_t(\cdot))\mathrm{d}W_t - \frac{1}{2}\int_0^s |X_t^{-\theta}(\gamma\mu_t(\cdot) - \gamma\nu_t(\cdot))|^2 \mathrm{d}t \right\}, \quad s \in [0, T]$$

Idea of Proof-Log-Harnack inequality

- For any $n \geq 1$, $(R_{s\wedge\tau_n})_{s\in[0,T]}$ is a martingale and Girsanov's transform yields that $(\tilde{W}_{s\wedge\tau_n})_{s\in[0,T]}$ is a one-dimensional Brownian motion under $\mathbb{Q}_T^n = R_{T\wedge\tau_n}\mathbb{P}$. So, we arrive at

$$
\begin{aligned}
&\mathbb{E}(R_{s\wedge\tau_n}\log(R_{s\wedge\tau_n}))\\
&\leq \frac{1}{2}\mathbb{E}^{\mathbb{Q}_T^n}\int_0^s X_t^{-2\theta}|\gamma\mu_t(\cdot)-\gamma\nu_t(\cdot)|^2\mathrm{d}t\\
&\leq \frac{1}{2}\gamma^2(\mathbb{E}|X_0-Y_0|)^2\tilde{\Gamma}.
\end{aligned}
$$

By martingale convergence theorem and $\lim_{n\to\infty}\tau_n=\infty$, we have $\mathbb{E}R_s = 1$ which implies that $\{R_s\}_{s\in[0,T]}$ is a martingale. Moreover, Fatou's Lemma yields

$$
\mathbb{E}(R_s\log(R_s)) \leq \frac{1}{2}\gamma^2(\mathbb{E}|X_0-Y_0|)^2\tilde{\Gamma}.
$$

Idea of Proof-Log Harnack inequality

- By coupling by change of measure for the classical SDE, there exists $\{\xi_t\}_{t\in[0,T]}$ and $\{Y_t\}_{t\in[0,T]}$, $\{\bar{W}\}_{t\in[0,T]}$ with $\bar{W}_t = \tilde{W}_t + \int_0^t \xi_s \mathrm{d}s$ such that $\bar{\mathbb{Q}}_T = \bar{R}_T R_T \mathbb{P}$ is a probability measure, where

$$\bar{R}_t = \exp\left\{-\int_0^t \xi_s \mathrm{d}\tilde{W}_s - \frac{1}{2}\int_0^t |\xi_s|^2 \mathrm{d}s\right\}, \quad t\in[0,T].$$

Moreover, it holds $\mathscr{L}_{Y_t}|\bar{\mathbb{Q}}_T = \nu_t, t\in[0,T]$ and $\bar{\mathbb{Q}}_T$-a.s. $X_t = Y_t, t\in[0,T]$ and

$$\mathbb{E}^{\bar{\mathbb{Q}}_T}\log(\bar{R}_T) = \frac{1}{2}\mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^T |\xi_s|^2 \mathrm{d}s \le \frac{(1-\theta)(\delta-\frac{\theta}{2})\mathbb{W}_{2,\rho}(\mu_0,\nu_0)^2}{(\mathrm{e}^{2(1-\theta)(\delta-\frac{\theta}{2})T}-1)}.$$

Noting that $\bar{\mathbb{Q}}_T$-a.s. $X_t = Y_t, t\in[0,T]$, we have

$$\mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^s X_t^{-2\theta}|\gamma\mu_t(\cdot)-\gamma\nu_t(\cdot)|^2\mathrm{d}t = \frac{1}{2}\mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^s Y_t^{-2\theta}|\gamma\mu_t(\cdot)-\gamma\nu_t(\cdot)|^2\mathrm{d}t.$$

So, we have

$$\mathbb{E}^{\bar{\mathbb{Q}}_T}\log(\bar{R}_T R_T) \le \mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^T |\xi_s|^2\mathrm{d}s + \mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^s Y_t^{-2\theta}|\gamma\mu_t(\cdot)-\gamma\nu_t(\cdot)|^2\mathrm{d}t.$$

So, we complete the proof.

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Thank You